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ABSTRACT

Part I of this guide develops sets and set operations: listing and describing sets, characteristics of sets, universal empty sets, unions and intersections of sets, properties under set operations, differences and complements of sets, Venn diagram interpretations, set notation, ordered pairs, Cartesian products, and equivalent sets. Part II deals with logic in terms of truth values of statements, conditional statements, conjunctions, disjunctions, equivalences, and negations, logic symbols, truth tables, determining logical validity of arguments (including Modus Tollens, Modus Tollendo Ponens, basic syllogisms), universal and existential quantifiers, symbols of quantification, and indirect proof (contradiction). The guide includes basic student exercises, challenges, suggested teacher questions, and pictorial illustrations of topics. (JW)

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SETS AND LOGIC

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SETS AND LOGIC

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THIRTEEN COLLEGE CURRICULUM PROGRAM
in conjunction with
CURRICULUM RESOURCES GROUP OF ISE

ABOUT THE INSTITUTE FOR SERVICES TO EDUCATION

The Institute for Services to Education was incorporated as a non-profit organization in 1965 and received a basic grant from the Carnegie Corporation of New York. The organization is founded on the principle that education today requires a fresh examination of what is worth teaching and how to teach it. ISE undertakes a variety of educational task , working cooperatively with other educational institutions, under grants from government agencie. and private foundations. ISE is a catalyst for change. It does not just produce educational materia's or techniques that are innovative; it develops, in cooperation with teachers and administrators, procedures for effective installation of successful materials and techniques in the colleges.

ISE is headed by Dr. Elias Blake, Jr., a former teacher and is staffed by college teachers with experience in working with disadvantaged youth and Black youth in educational settings both in predominantly Black and predominantly white colleges and schools.

ISE's Board of Directors consists of persons in the higher education system with histories of involvement in curriculum change. The Board members are:

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ABOUT THE THIRTEEN-COLLEGE CURRICULUM PROGRAM

From 1967 to the present, ISE has been working cooperatively with the Thirteen-College Consortium in developing the Thirteen-College Curriculum Program. The Thirteen-College Curriculum Program is an educational experiment that includes developing new curricular materials for the entire freshman year of college in the areas of English, mathematics, social science, physical science, and biology and two sophomore year courses, humanities and philosophy. The program is designed to reduce the attrition rate of entering freshmen through well thought-out, new curricular materials, new teaching styles, and new faculty arrangements for instruction. In addition, the program seeks to alter the educational pattern of the institutions involved by changing blocks of courses rather than by developing single courses. In this sense, the Thirteen-College Curriculum Program is viewed not only as a curriculum program with a consistent set of academic goals for the separate courses, but also as a vehicle to produce new and pertinent educational changes within the consortium institutions. At ISE, the program is directed by Dr. Frederick S. Humphries,

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In addition, Miss Patricia Parrish serves as general editor of the curriculum materials as well as an Administrative Assistant to the Director. Mrs. Joan Cooke is Secretary to the Director.

The curriculum staff is assisted in the generation of new educational ideas and teaching strategies by teachers in the participating colleges and outside consultants. Each of the curriculum areas has its own advisory committee, with members drawn from distinguished scholars in the field but outside the program.

The number of colleges participating in the program has grown from the original thirteen of 1967 to nineteen in 1970. The original thirteen colleges are:

Alabama A and M University	Huntsville, Alabama
Bennett College	Greensboro, North Carolina
Clark College	Dallas, Texas
Florida A and M University	Atlanta, Georgia
Jackson State College	Tallahassee, Florida
Lincoln University	Lincoln University, Pennsylvania
Norfolk State College	Norfolk, Virginia
North Carolina A and T State University	Greensboro, North Carolina
Southern University	Baton Rouge, Louisiana
Talladega College	Talladega, Alabama
Tennessee State University	Nashville, Tennessee
Voorhees College	Denmark, South Carolina

A fourteenth college joined this consortium in 1968, although it is still called the Thirteen-College Consortium. The fourteenth member is:

Mary Holmes Junior College	West Point, Mississippi

In 1970, five more colleges joined the effort although linking up as a separate consortium. The members of the Five-College Consortium are:

Elizabeth City State University	Elizabeth City, North Carolina
Langston University	Langston, Oklahoma
Southern University at Shreveport	Shreveport, Louisiana
Saint Augustine's College	Raleigh, North Carolina
Texas Southern University	Houston, Texas

In 1971, eight more colleges joined the curriculum development effort as another consortium. The member schools of the Eight College Consortium are:

Alcorn A and M College	Lorman, Mississippi
Bethune-Cookman College	Daytona Beach, Florida
Grambling College	Grambling, Louisiana
Jarvis Christian College	Hawkins, Texas
LeMoyne-Owen College	Memphis, Tennessee
Southern University in New Orleans	New Orleans, Louisiana
University of Maryland, Eastern Shore	Princess Anne, Maryland
Virginia Union University	Richmond, Virginia

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The Carnegie Corporation
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(AL)-Alabama A & M; (BE)-Bennett; (BI)-Bishop; (CL)-Clark; (EC)-Elizabeth City; (FL)-Florida A & M; (JA)-Jackson State; (LA)-Langston; (LI)-Lincoln; (MH)-Mary Holmes; (NF)-Norfolk State; (NC)-N. Carolina A & T; (SB)-Southern U.; (SS)-Southern at Shreveport; (SA)-St. Augustine; (TA)-Talledega; (TE)-Tennessee; (TS)-Texas Southern; (VO)-Voorhees; (SAC)-Southern Association; (ISE)-Institute For Services to Ed.

SETS AND LOGIC

PART I

SETS AND LOGIC

Introduction

Why are sets important in mathematics? The concept of sets pervades all of mathematics and is used in many of our daily activities, as well. Think of the many times you refer to a pen and pencil set, a set of dishes, a set of encyclopedias, or in some other way identify a collection of items as a set.

In mathematics we use such statements as:

Sets of numbers in Arithmetic

Sets of solutions of equations in Algebra

Sets of points in Geometry

Mathematicians consider sets of elements to be a unifying theme throughout mathematics.

I. The Idea of Sets

Today mathematicians and scientists think and talk in terms of sets. The whole of pure mathematics may be reduced to statements about sets. The set idea is used tacitly every time objects are classified by their properties. Saying that something has a particular property is equivalent to saying it belongs to the set of everything having that property. The idea is a simple yet powerful mathematical tool. The notion of a set is the mental construct obtained by regarding several discrete things as constituting a single whole, a collection. The human mind arbitrarily considers together certain things and regards the collection itself as a new kind of thing.

Undefined Terms:

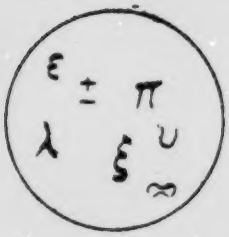
Using the unabridged dictionary lead the class to discover the circularity of dictionary definitions. To avoid circularity it is necessary to begin by choosing a small number of words accepted without definition. The word "set" is such an undefined word. So is the relation "is an element of." Note: Finding a model must be distinguished from giving a definition.

These are pictures of sets:

(1)

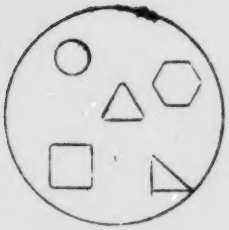
the set of vowels

(2)



a set of symbols

(3)



a set of geometric figures

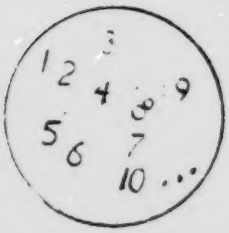
(4)

a set of flowers

(5)

an empty set

(6)



the set of natural numbers

(7) ← · · · · · · · → a set of points

These are examples of sets where each element is listed.

(1) M = {Washington, Oregon, California, Alaska}.

(2) N = {5, 10, 15, 20, . . . ,100, 105}.

(3) C = {a, b, c, d, e, . . . , x, y, z}.

(4) B = {peach, pear, prune, plum, pineapple}.

(5) A = {Mass., Seven, Liberty, D, Hope Diamond}.

(6) K = {January, March, May, July, August, October, December}.

(7) E = {Adams, Johnson, University, The Liberty Bell}.

These are examples of sets where the elements are described:

(1) Y = the set of all students in this room with an Afro.

(2) W = the set of months of the year whose names begin with the letter D.

(3) Z = the set of all living Americans at this moment.

(4) S = the set of United States Senators.

(5) I = the set of all points which lie in the interior of this 1 cm. square

(6) D = the set of days of the week whose names begin with the letter L.

Be sure the students recognize the two standard ways of indicating the members of a set: (1) listing all the members (enumeration) and (2) giving a rule for determining the members (set abstraction). Sets N and C of the exercise are a Hybrid of (1) and (2)---enough enumeration so that the rule can be readily induced.

Forming Sets

Instructions:

1. Give the students five or six boxes of attribute blocks and loops.
2. You may suggest that they work in groups.
3. Have them assemble the members of the following sets. The teacher should point out that the usual notation used for this is of the form, {x|x is blue}.
    a. The set of all blocks that are blue and large.
    b. The set of all blocks that are quadrilaterals.
    c. The set of all blocks that are not square and large.
    d. The set of all blocks that are diamonds or small.
    e. The set of all blocks that are red and not circular.
    f. The set of all blocks that are not diamonds and not large.

g. The set of all blocks that are blue or large, and not square.

h. The set of all blocks whose surface area is the same on all sides.

i. The set of all blocks that are not yellow, and six sided.

j. The set of all blocks that are not polyhedra.

k. The set of all blocks that are green and three sided.

A set A is said to be well defined provided that for any element d, the answer to the question, "Is d an element of A?" is either a "yes" or a "no", but not both.

In discussing general properties of sets, it is convenient to use capital letters, A, B, C, ..., to denote specific sets, and lower case letters, a, b, c, ..., to denote the elements in sets. These elements may be of diverse kinds. They are called elements or members of the set to which they belong. If d is an element of set A, we indicate this fact by saying d is an element of A (denoted by $d \in A$).

Challenges:

Consider the following sets:

G = {Plum, Pineapple, Peach, Pear, Prune}.

M = {3, π, capitalism, Tower of Babel, 9, 10, 11, Washington Monument},

N = the United States Senators.

P = the set of all living Americans.

T = {0}.

R = {a, b, c, d, e, . . ., x, y, z}.

W = {2, 4, . . . }.

Q = the set of all points which lie in the interior of this 1 cm. square.

S = { }.

1. What are the members of the set G?
2. What are the elements of set M?
3. Is 12 a member of set M?
4. Name three elements of N?
5. Name the elements of set Q?
6. Peach $\in$ G, but Banana $\notin$ G. 5 $\in$ M, but 2 $\notin$ M. A $\in$ R, K $\in$ R, boy $\notin$ R.

True or False: 8 $\in$ M _____

G $\in$ P _____

Approximately 200 million people belong to N _____

1/2 $\in$ M _____

Plum $\in$ G _____

J ε R _____

7. Is plum a member (or element) of set G? Express your answer symbolically.

8. Explain how the following sets differ:

$I' = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$

$I'' = \{ 1, 2, 3, \ldots, 10 \}$

$I = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots \}$

9. Is Benjamin Franklin a member of P? Express your answer symbolically.

10. What are the next three elements of set W?

11. Does set T = set S?

12. Describe a set which has at least ten elements; at least a hundred elements; at least a thousand elements; at least a million elements. Can you think of a set with no elements whatsoever?

13. How many members are in set R? How many members are in set N?

14. How many members are in set Q?

15. Which ones of the sets G, M, N, P, R and Q are finite? Which ones are infinite?

16. Give another example of an infinite set.

17. Is the set of all the grains of sand on the beaches of the United States at this very second finite or infinite?

18. Describe each set below:

    (a) A = { 1, 3, 5, 7, 9 }.

    (b) T = { Tuesday, Thursday }.

    (c) P = { bus, plane, train, automobile }.

    (d) E = { 1 }.

    (e) F = { 5, 10, 15, . . . }.

19. How many synonyms can you think of for the term set?

Instructions

1. Place a green loop inside a black loop.
2. Place a small red triangle and a small red square inside the green loop.
3. Place a large green triangle and a large green square inside the black loop but not in the green loop.

Questions

1. List the members of the set of things inside the green loop.
2. List the members of the set of things inside the black loop. (Be sure the students recognize that everything in the green loop is also in the black loop).
3. Are all the members of the set of things inside the green loop also members of the set of things inside the black loop? Explain.

At this point the instructor should introduce the words subset, proper subset, and equal sets and give formal definitions.

Definition 1: A set S_1 is said to be a subset of a set S, written $S_1 \subset S_2$, if every element of S_1 is also an element of S_2.

Definition 2: A set S_1 is said to be equal to Set S_2, written $S_1 = S_2$, provided that $S_1 \subset S_2$ and $S_2 \subset S_1$.

Definition 3: A set S_1 is said to be a proper subset of a set S_2 if and only if $S_1 \subset S_2$ and $S_1 \neq S_2$.

Questions:

1. If you move the green loop so part of it projects outside the black loop with the red objects still within the black loop, is the set of objects within the green loop still a subset of the objects within the black loop? (The answer is yes, because sets are defined by their members, not by their loops.)
2. Is the set of triangles a subset of the set of objects within the black loop?
3. Give the members of another subset of objects within the black loop.
4. Is the set of green objects a subset of the set of objects within the black loop?

Given a set K = { a, b, c, d, }. How many subsets of K have

(a) exactly 1 element?

(b) exactly 2 elements?

(c) exactly 3 elements?

(d) exactly 4 elements?

(e) no elements?

Question: What is the total number of subsets?

Let's record some of this data in a table.

number of elements in the subset	number of subsets having exactly the following number of elements: 0 1 2 3 4 5 6 7 . . .
0	
1	
2	
3	
.	
.	
.	
n	$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}$ . . .

Guided Questions:

1. Suppose we have a set containing no elements, i.e. the empty set. How many subsets can be formed which contain no elements?
2. Given a set containing <u>one</u> element, how many subsets have no elements? 1 element?
3. Given a set containing 2 elements, how many subsets have no elements? 1 element? 2 elements? etc.

Continue this line of questioning for sets containing 3, 4, and perhaps 5 elements. Tell students to examine the table and attempt to supply requested data for a set containing 5 or 6 elements.

Question: What do you notice about the array of numbers in the table?

Question: How can the total number of subsets of any given set be determined?

Challenges:

1. Suppose a given line in the array is:

   1 9 36 84 126 126 84 36 9 1

   then the next line is:

2. Given line:

   1 11 55 165 330 462 462 330 165 55 11 1

   then the next line is:

3. (a) How many subsets can be formed from a set containing 7 elements? 8 elements? n elements? 2n elements?

   (b) How many non-empty subsets can be obtained from a set containing n elements?

II. Universal Sets and Empty Sets

Sometimes it is convenient to talk about a set that includes all the elements of all sets which we shall discuss.

Suppose we close our minds to all numerals except those in set A = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }. The set consisting of all the elements under consideration will be designated by the symbol $\cup$. It is called the universe. Now consider these sets:

B = { 0, 2, 4, 6, 8}.

C = { 1, 3, 5, 7, 9 }.

D = { 0, 1, 3, 4 }.

Questions:

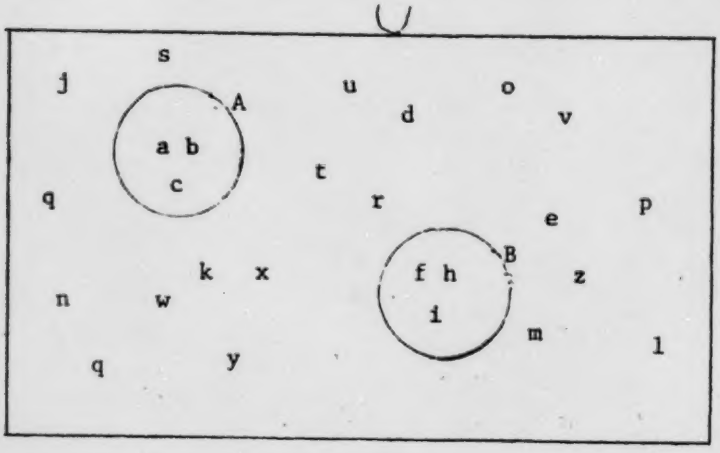
1. How is set C related to set B?
2. How is set B related to set D?
3. How are sets B, C and D related to set A?

Definition: If all the sets under consideration are subsets of some one set we shall call this "largest" set the universal set of the discussion.

Example: If $\cup$ = { a, b, c, d, e, ... z }

A = { a, b, c } and B = { f, h, i} then $A \subset \cup$ and $B \subset \cup$.

(See figure below.)



A set is often best defined by stating a property which every element in the collection and no other has. However, a property might be defined which none of the elements considered could have. Consider the following sets, then see if you can form some conclusion about them.

1. The set consisting of all the four-dollar bills in circulation in the United States.
2. The set of all rectangles with five sides.
3. The set of all odd numbers divisible by 2.
4. The set of all math books on an empty shelf.
5. The set of all integers less than five but greater than thirteen.

Do you see any particular property of the set that is common to all five of the above sets? Would it be correct to say that each of the above sets equals zero? Do you think the words null, nothing, empty and zero each mean something different from the other?

It turns out that each of the five above sets exists, but is empty, i.e. has no members. The empty set has the property zero, but the empty set is not itself zero. An empty set can be symbolized by curly brackets with nothing between them {}, or by ϕ .

There is only one empty set. Since two sets were defined to be equal if they consisted of the same elements, any two empty sets are equal. They both have no elements. The empty set is a subset of every set, and every set is a subset of itself.

Challenge: Which of these sets is the null or empty set?

$A = \{ x | x \text{ is a letter before the letter a in the alphabet} \}$.

$B = \{ x | x^2 = 16 \text{ and } 2x = 6 \}$.

$C = \{ y | y \neq y \}$.

$D = \{ x | x + 7 = 7 \}$.

III. New Sets From Old (Union and Intersection)

Let us continue our discussions of sets by playing this game. Here are the rules:

1. Class is divided into two teams.
2. Four coins are to be used in this game (a penny, a nickel, a dime, and a quarter). Let's call the set of coins

A = { p, n, d, q }. Ask the students to list the subsets of A: {}, {p}, {n}, {d}, {q}, {p, n}, {p, d}, {p, q}, {n, d}, {n, q }, {d, q }, {p, d, n }, {p, d,q}, {p, n, q}, {n, d, q}, {p, n, d, q}.

3. List all of the amounts that can be formed from the coins above.

4. The object of the game is to accumulate the most money.

   (a) Each play must be followed by a play which must contain one or more coins in common with the previous play.

   (b) A play may not be used more than once during a game. The penalty for repeating a play might be the lose of a turn or adding the amount to the opposing team. The game is over only when all plays have been used or a player surrenders. Example: 35¢ may be followed by 15¢ because they have a dime in common. After a winner has been declared, the class should examine the data. The instructor should now give a formal definition for intersection.

Definition: The intersection of two sets A and B written $A \cap B$, is that set consisting of those elements which are in both A and B.

Symbolic Definition: $A \cap B = \{ x | x \in A \text{ and } x \in B \}$,

Example 1 : If $A = \{ a, b, c \}$ and $B = \{ a, c, e, g \}$ then

$A \cap B = \{ a, c \}$.

Example 2 : If $A = \{ a | a \in I, a > 0 \}$ and $B = \{ b | b \in I, b < 5 \}$,

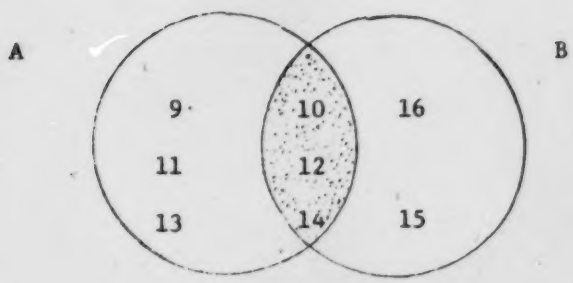
then $A \cap B = \{ 1, 2, 3, 4 \}$.

To help the students visualize operations with sets the Venn diagram can be used. In a Venn diagram we let the elements of a set be represented by points in a plane region.

Example: If $A = \{ 9, 10, 11, 12, 13, 14 \}$ and $B = \{ 10, 12, 14, 16 \}$,

then $A \cap B = \{ 10, 12, 14 \}$.

The shaded area represents $A \cap B$.



Suppose set A = { a, b, c } and its subsets B = {a}, C = {b}, D = {c}, E = { a, b }, F = { a, c }, G = { b, c }, H = { a, b, c }, and I = ϕ. Make a table using the operation of $\cap$ and let the class fill in the data. The instructor can use the table to review or introduce equal sets, disjoint sets, etc.

$\cap$	B	C	D	E	F	G	H	I
B	B	I	I	B	B	I	B	I
C	I	C	I	C	I	C	C	I
D	I	I	D	I	D	D	D	I
E	B	C	I	E	B	C	D	I
F	B	I	D	B	F	D	F	I
G	I	C	D	C	D	G	G	I
H	B	C	D	E	F	G	H	I
I	I	I	I	I	I	I	I	I

Find the following and symbolize with diagrams.

1. The intersection of a set and one of its proper subsets is always the proper subset.
2. The intersection of a set with itself is the set itself.
3. The intersection of a set and the null set is the null set.

4. The intersection of two disjoint sets is the null set.
5. The operation of intersection is commutative.
6. The operation of intersection is associative.
7. The subsets of a set is a closed set with the operation of intersection.

See if students are able to give informal proofs of any of these seven statements.

Let set A = { b, c } and set B = {a}. Put the members of sets A and B into a new set. We get { a, b, c }. Let us call this set C. It is called the union of A and B. The notation for this is $A \cup B = C$.

Exercises:

1. If D = { w, Biafra, 17 }, E={ freedom, r, Fisk }, what is $D \cup E$?
2. If M = the set of all men and W = the set of all women, what is $M \cup W$?
3. If G = { a, b, 2, 4 } and H = { b, c, 4, 5 }, what is $G \cup H$?

Rembering the fact that no element need be listed twice in a set, let us take the set A = { a, b, c }, and its subsets B = {a}, C = {b}, D = {c}, E = {a, b} F = {a, c} , G ={ b, c }, H = {a, b, c}, $I = \phi$, construct their unions and fill out the following table.

Example: B = {a} and C = {b}. Thus B ∪ C = {a, b}

∪	B	C	D	E	F	G	H	I
B	B	E	F	E	F	H	H	B
C	E	C	G	E	H	G	H	C
D	F	G	D	H	F	G	H	D
E	E	E	H	E	H	H	H	E
F	F	H	F	H	F	H	H	F
G	H	G	G	H	H	G	H	G
H	H	H	H	H	H	H	H	H
I	B	C	D	E	F	G	H	I

The teacher will probably find the students can now give a formal definition for union and the formal symbol.

Definition: The union of two sets A and B, written $A \cup B$, is that set consisting of elements of A or B or both A and B. $A \cup B$ is $\{x | x \in A \text{ or } x \{B\}$.

Example: If A = {1, 2, 3, 4, 5} and B = {3, 5, 7, 9} then $A \cup B$ = {1, 2, 3, 4, 5, 7, 9}.

The Venn diagram will help in visualizing the union of the sets A and B.

Ask the students to make a list of observations from the table. They should discover the following:

1. The union of a set and one of its proper subsets is always the set.
2. The union of a set with itself is the set itself.
3. The union of a set and the null set is the set.
4. The union of two non-empty disjoint sets is not the null set.
5. The operation of union is commutative.
6. The operation of union is associative.
7. The subsets of a set is a closed set with the operation of union.

If the class is familiar with the distributive property of real numbers, ask them to apply this principle to sets using the the operations of union and intersection. They should discover two distributive laws with sets: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercises:

1. If A is the set of all even whole numbers and B is the set of all odd whole numbers, what is the set $A \cup B$? What set is $A \cap B$?

2. Let A and B be sets. Give conditions under which:

   (a) $A \cup B = A$.
   (b) $A \cap B = A$.
   (c) $A \cup \phi = A$.
   (d) $A \cap \phi = A$.
   (e) $A \cap B = \phi$.
   (f) $A \cup B = \phi$.
   (g) $A \cup B = B \cup A$.
   (h) $A \cup B = A \cap B$.

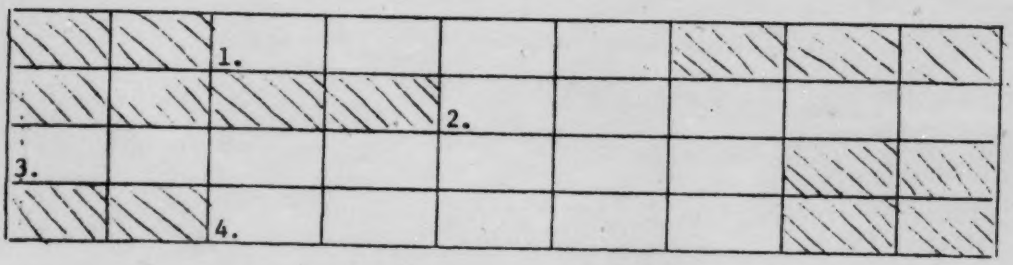
3. If $R = \{ x | x \in I \text{ and } x + 11 \geq 20 \}$ and $S = \{ y | y \in I \text{ and } Y - 11 < 101 \}$ find $R \cup S$ and $R \cap S$.

4. Given the set $A = \{p, n, d, q\}$ find the subsets of A. Make two tables from the subsets of A using the operations intersection and union.

5. Is it possible for $X \cup Y$ to be the same set as X? Explain.

6. Is $S \cup T$ the same set as $T \cup S$ for all sets S and T?

7. Using only intersection and union, you can work this puzzle. Each resulting set of letters will need to be unscrambled to make a word. The four words read in order will make a familiar statement.



Across:

Across:

1. A = {a, h, G, s, d, o, p, o}.
   B = {r, d, f, o, G, l, o}.
   $A \cap B =$ _____.

2. C = {f, e, g}.
   D = {i, r}.
   $C \cup D =$ _____.

3. E = {h, i, s, e, m, r, a, q, C, l}.
   F = {w, C, n, l, x, h, r, e, i, p, a}.
   $E \cap F =$ _____.

4. G = {n, B, w}.
   H = {r, o}.
   $G \cup H =$ _____.

IV. Differences and Complements of Sets

Whenever we consider more thar one set of elements, it is possible that some elements belong to one of the sets and not any of the others. Also some elements may belong to all of the several sets, while there may be elements in the universe which do not belong to any of the sets.

Several other ideas may be developed from these relationships.

Suppose we consider the male population of a small rural high school with 120 male students as our universal set. Also consider the following:

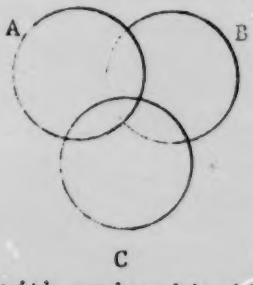
A = {Bob, Ed, James Henry, Roy, Tom} : the basketball team.

B = {James, Roy, Sam, Fred, Al} : the track team.

C = {Lou Dave, Roy, Fred, Bob} : the tennis team.

Considerations:

1. What team (s) does Bob play on?
2. Are there any boys who play on all three teams?
3. How many boys do not play on any team?
4. In the following diagram place the initials of each boy, once each in the appropriate region.



5. Form a new set, D, with membership limited to those boys who play basketball but do not run track, i.e., D = {?}.

This set D may be described as the difference between sets A and B, denoted by A - B. By definition then, A - B is the set of objects which belong to A but which do not belong to B. In Symbols: $A - B = \{x | x \in A \text{ and } x \notin B\}$

Considerations:

1. What elements belong to B - A?

2. In this diagram dot the area representing A - B.
3. Show B - A by a diagram.

Suppose we let M represent the set of male students who are not members of any athletic teams, then

M = {x|x is a male student and is not a member of an athletic team}

Set M may be labeled in a different fashion, i.e., M = $\overline{D}$ where $\overline{D}$ is called the complement of D, i.e.

Challenges: $\overline{D}$ = {x|x $\notin$ D, x $\in$ $\cup$}

1. Construct diagrams to show the following. Shade the indicated parts.

   (a) A - B (c) $\overline{A}$ (e) $\overline{(A - B)}$

   (b) B - A (d) $\overline{B}$ (f) $\overline{(B - A)}$

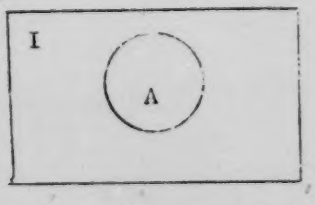
2. Suppose A $\subset$ B. Construct diagrams to show relatiohships (a)-(f) above. Do the same if A and B have no elements in common.

3. Were you able to show all of the diagrams suggested above? Why?

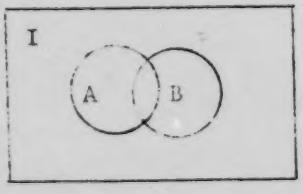
*4. Prove that B - A is a subset of $\overline{A}$.

*5. Prove that B - A = B $\cap$ $\overline{A}$.

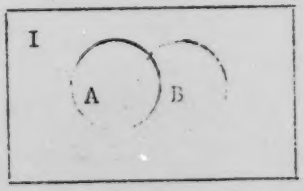
6. Use dotting to illustrate the following:



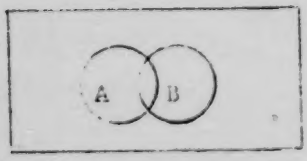
$\overline{A}$



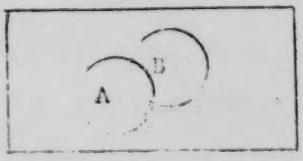
A - B



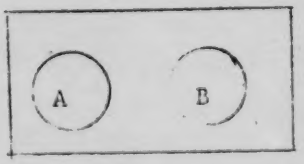
B - A



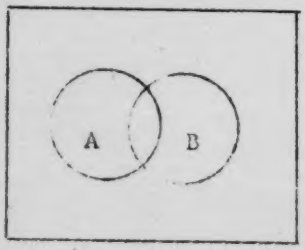
$\overline{(A \cap B)}$



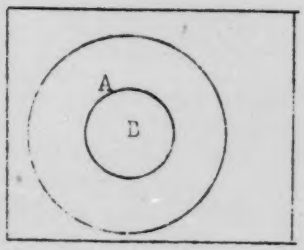
$\overline{(A \cup B)}$



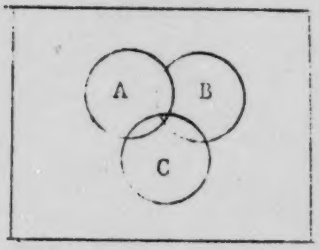
$\overline{A} \cap \overline{B}$



$\overline{A} \cap \overline{B}$



$\overline{A}$



$\overline{(A \cap B \cap C)}$

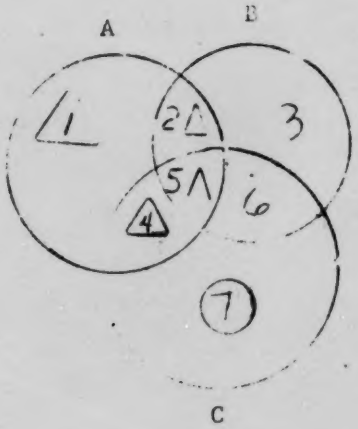
7. Let U = {a, b, c, d, e} ,

   A = {a, b, d} and B = {b, d, e}

   Find:

   1. B - A
   2. $\overline{B}$
   3. $\overline{A} \cap B$
   4. $A \cup \overline{B}$
   5. $\overline{A} \cap \overline{B}$
   6. $\overline{B} - \overline{A}$

Review Problems



Let 1 be a large red triangle,

2 be a small green triangle,

3 be empty,

4 be a large black triangle,

5 be a small black triangle,

6 be empty and

7 be a large blue circle.

Below are several statements true of this Vern Diagram. The teacher can give any one of the equal quantities and ask the students to give the other(s), or point to a region and ask for it to be characterized in as many ways as students can think of.

1. $A = \{x|x \text{ is a non-blue}\}$
   $= \{x|x \text{ is a } \Delta\}$
   $= \{x|x \text{ is a non-circle}\}$
   $= \upsilon - \{x|x \text{ is a circle}\}$,
2. $B = \{x|x \text{ is small}\} = \{\overline{x|x \text{ is large}}\}$,
3. $C = \{x|x \text{ is not red } \underline{\text{and}} \text{ x is not green}\} = \{x|x \text{ is black } \underline{\text{or}} \text{ x is blue}\}$,
4. Explain $B \supset A$.
5. $A - B = \{x|x \text{ is a large } \wedge\}$.
6. $A \cup B = \{x|x \text{ is a } \wedge\} = A$.
7. $C - A = C - (A \cup B) = \{x|x \text{ is blue}\} = \{x|x \text{ is a circle}\}$
   $= \{\overline{x|x \text{ is a } \wedge}\}$.

8. $B - A = \{\ \} = \phi$.

9. $A \cap B = \{x | x$ is a small $\wedge\}$.

10. $A \cap C = \{x | x$ is black$\}$.

11. $(A \cap B) \cup (A \cap C) = \{x | x$ is a non-red $\wedge\}$.

12. $(A \cup C) - B = \{x | x$ is a large black $\wedge\}$.

13. $C \cap B = \{x | x$ is a small black $\wedge\}$.

14. $A \cup C =$ universe $= \{x | x$ is a $\wedge$ <u>or</u> x is a circle$\}$.

15. $(C \cap B) - A = \{\ \} = \phi$.

V. <u>Cartesian Products</u>

Order is very important: it is standard to put your underwear on before your slacks. If you put your pants on first you would be a ridiculous sight on the street. — You might even be arrested. Since a set is fully determined by its members the order is not important, ie. {shorts, pants} = {pants, shorts}. So ordered pairs are invented. They are indicated by using regular parentheses rather than the braces. Thus we can use (shorts, slacks) to represent that the shorts go on first. This set with two members is called an ordered pair.

Suppose we had a set of boys, B = {Jackie, James, Tony, Willie, Chris}, and a set of girls, G = {Jackie, Mary, Willie}.

Here no two boys and no two girls have the same name. But we have a couple of cases of a boy and a girl having the same name. Thus, if we say Jackie and Willie have a date we wouldn't know whether it is the boy Jackie and the girl Willie, or the girl Jackie and the boy Willie. A very simple way to remove the ambiguity is to agree always to name the boy first in the couple. We can use our new ordered pair notation. Thus the ordered pair (Jackie, Willie) indicates the boy

Jackie and the girl Willie are having the date. What does (Willie, Jackie) indicate using this agreement? It certainly is a different couple. How can we indicate that the boy Chris and the girl Willie are having a date?

When are two ordered pairs equal? This means when do two ordered pairs represent the same couple? You can readily see that (Jackie, Willie) = (Mr. Harris, Miss Johnson) if and only if Jackie = Mr. Harris, and Willie = Miss Johnson. In general (a, b) = (c, d) if and only if a = c and b = d.

Questions:

1. What could we conclude from (Willie, Mary) = (Mr. Blair, Miss Clark)?
2. Given g = h and j = k which of the following can we conclude:

   (a) (g, h) = (j, k).

   (b) (h, j) = (g, k).

   (c) (k, g) = (h, j).

   (d) (j, h) = (k, g).

(ans: b, d)

Suppose now that our five boys and three girls are all interested in "playing the field", i.e. in having dates: each of the boys with each of the girls. We want to list all the different pairing for dates that are possible. Let us start with all the dates the boy

Jackie could have.

B = {Jackie, Tony, Chris, James, Willie}

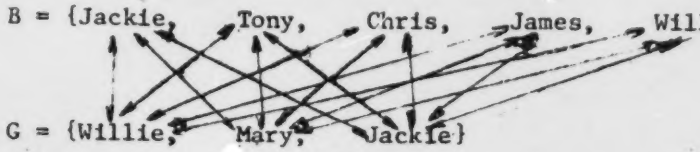
G = {Willie, Mary, Jackie}

So the set of dates he could have are

(Jackie, Willie), (Jackie, Mary), (Jackie, Jackie)

And repeat for Tony, Chris, James, and Willie:

B = {Jackie, Tony, Chris, James, Willie}



G = {Willie, Mary, Jackie}

1. List the set of possible ordered pairs.
2. How many different couples can be formed?
3. How many different dates oculd each boy have? Each girl?

This process of forming a new set by the ordered pairing of the members of two given sets is called forming their Cartesian Product. The first member of each ordered pair is a member of the first, the second member of the ordered pair is a member of the second set; and all such pairs possible are included. The Cartesian product of B and G is indicated by B X G and called "B cross G". It is a set whose members are ordered pairs.

1. Is B X G = G X B? (No. (Tony, Mary) is in B X G but not in G X B.)
2. C = {2, 4, 6}; D = {a, b, c, d}. Find C X D.

3. E = {1, 2, 3, 4}; F = {2, 4, 6}. Find E X F.

4. Can you now give a rigorous definition for the Cartesian Product of two sets? (If B and G are sets, then set B X G, called the Cartesian Product of B and G, is the set of all ordered pairs such that the first member of each pair is an element of B and the second member is an element of G.)

5. If A = {2, a, #}, find A X A.

6. When does A X B = B X A?

7. When is A X B ≠ B X A?

8. Can we say in general that Cartesian Products are commutative?

9. Do you smell any parallelism between ordinary multiplication and Cartesian Products?

10. A penny and a dime are tossed together. The set of possible outcomes for the dime are (using "H" for "heads," "T" for "tails") D = {H, T}. What is the set P of possible outcomes for the penny? Write out the set P X D. What does P X D represent? (Set of all possible outcomes for the dime and penny thrown once together.)

11. Suppose a red die and a white die are tossed together. Give the set of all possible outcomes of the one toss of the two dice.

12. A penny, a nickel, and a dime are thrown together. Give the set of all possible outcomes. (a set of ordered triples, P X N X D = {(H, H, H), (H, H, T), ...}.)

VI. Equivalent Sets

An Anectdote

Back when no one knew how to read or write a shepherd had a real problem when he wanted to be sure that his entire flock had returned and none was lost in the field. It was a great day when some inventive lad thought of an ingenious solution to this problem. He took a pile of pebbles to the gate where the sheep went out to pasture. He placed one pebble in the sack for each sheep. At dusk, when they returned, he reversed the procedure and removed a pebble for each sheep. If none of the pebbles were left in the sack the shepherd knew that all had returned. When there were pebbles left in the sack, he knew how many sheep were missing.

Let us investigate the underlying idea. Let us call the collection of pebbles set P, and that of the sheep as set S. The action of putting one pebble for each sheep is a one-to-one matching of elements of two sets. For every pebble in P there corresponds a sheep in S and conversely for every sheep in S there corresponds a pebble in P. So if any pebble is left over when the sheep return, a sheep is known to be missing.

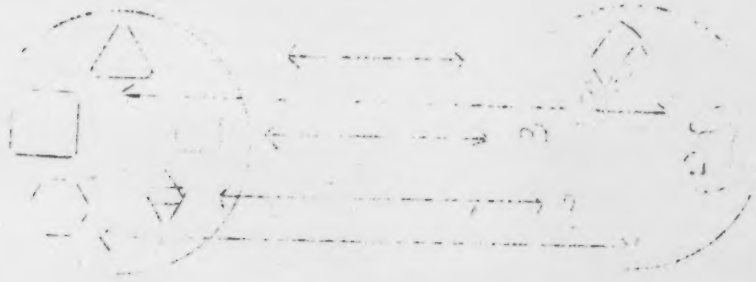
Definition: Given any two sets A and B. If for every element in A, there corresponds a unique element in B, and vise versa, we say that a one-to-one correspondence exists between the two sets. Whenever this is so, we say the two sets are equivalent.

The elements of the sets may be anything we choose. Example: the students in a class are one set and the corresponding chairs are another set. If there are as many chairs as students and as many students as chairs, then the two sets are equivalent. No extra elements should be left over in the matching.

Consider another example of two trays containing some articles. The articles in two trays form two separate sets. Let the first tray contain some geometrical figures, say triangles, rectangles, etc. And the second tray contain books, pencils, etc.

(Equivalent Sets)

The one-to-one correspondence is shown by double arrows. Similarly non-equivalent sets are those where one-to-one correspondence does not exist. As shown in the diagram



(Non-Equivalent Sets)

Consider now the sets whose elements are arithmetic numbers, then the equivalent and non-equivalent sets can be written as

$$A = \{1, 3, 5, 7\}$$
$$\uparrow\ \uparrow\ \uparrow\ \uparrow$$
$$\downarrow\ \downarrow\ \downarrow\ \downarrow$$
$$B = \{2, 4, 6, 8\}$$

Equivalent Sets

and

$$A = \{1, 3, 5, \ \}$$
$$\uparrow\ \uparrow\ \uparrow\ \uparrow$$
$$\downarrow\ \downarrow\ \downarrow\ \downarrow$$
$$B = \{2, 4, 6, 8\}$$

Non-equivalent Sets

If, however, the elements of the sets are algebraic numbers then we have

$$A = \{P, Q, R, S, T\}$$
$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$B = \{p, q, g, s, t\}$$

And

$$A = \{P, Q, R, S, T, U\}$$
$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$\{p, q, ?, s, \quad\}$$

Challenges

(a) If all the marriageable boys are taken as a set B and all the marriageable girls are taken as a set G, do you think the two sets are equivalent? Explain.

(b) If the voters form a set A and the ballots they cast form set B, under what condition will the two sets be equivalent?

(c) Is the set of wheels on a car equivalent to the set of members of a quartet?

(d) Is the set of the seven deadly sins equivalent to the set of days in the week?

(e) Is the set of operations consisting of adding 1, adding 2, and 3 equivalent to the set of operations at adding 2, and adding 4?

Mathematics is more than computing answers to routine problems. In mathematics we set up systems of axioms. With logic we test them for consistency and from them derive theorems by valid argument.

Both are defined in terms of possible truth values of statements? so we start with the study of the truth of simple sentences, then complex sentences in standard logical forms using truth tables. From these we can define validity and consistency, and set up a system of derivation guaranteeing validity.

True - False Statements

Ask the students which of these sentences can be assigned a truth-value of "true" or "false".

_____ 1. How old are you?

_____ 2. Hand me the key .

_____ 3. I wish the football season were over .

_____ 4. The sum of the interior angles of a triangle is 180°.

_____ 5. One of the grossest errors that we can make is to assume that we have exhausted our intelligence and imagination.

_____ 6. Travel to the moon and you will see many wondrous sights.

_____ 7. You should obey orders.

_____ 8. Biafra is far away .

_____ 9. I am hungry .

_____ 10. She is tired .

_____ 11. Curves don't come too sharp or hills too steep for this nimble car.

_____ 12. Where does your charity dollar go?

_____ 13. Children play.

_____ 14. Be as impartial as a traffic light.

_____ 15. You need to be careful.

_____ 16. If it is raining, then I shall drive you home.

_____ 17. A blaze of fireworks lights the medieval face of Heidelberg, Germany.

_____ 18. Memorials are not enough.

_____ 19. Our nation cannot afford to waste its best yound minds.

_____ 20. When the doors are opened.

_____ 21. Watch for the open date.

_____ 22. Time is on the side of right.

_____ 23. It seems to me that the purpose of law is to achieve justice.

_____ 24. The hours that I walked among our lost regions were among the most poignant of my life.

_____ 25. Every citizen is native born or naturalized.

_____ 26. Help!

What kind of sentences are capable of having a truth-value of "true" or "false"? The student should discover that the true-or-false sentences are declarative while those that cannot be thought of as either true or false are imperative, interrogative or exclamatory or are utterly meaningless, not even sentences.

Logical Statements

The purpose of this section is to determine the truth values of common logical sentences, namely, the implication (or conditionals), the conjunction, the disjunction, the equivalence, and negations.

Give the students the following statements or sentences. Ask them to determine whether the statements or sentences are true or false. When an argument arises over the truth value of a particular statement, students should be encouraged to seek a common agreement.

Truth and Falsity

_____ 1. 16 is an even number and 16 is an odd number.

_____ 2. If Richard Nixon is the U. S. President, then you are a monkey's uncle.

_____ 3. If you are a monkey's uncle, then Richard Nixon is the U. S. President.

_____ 4. This statement is true or this statement is false.

_____ 5. A polygon is a square only if it is a rectangle.

_____ 6. Today is Monday if and only if tomorrow is Tuesday.

_____ 7. Adam was woman if and only if Eve was a man.

_____ 8. $3 > 3$ and $9 > 7$.

_____ 9. $5 > 2$ or $5 < 3$.

_____ 10. $6 > 3$ and $9 > 11$.

_____ 11. If 2 is a factor of 8, then 3 is afactor of eight.

_____ 12. $2 + 3 = 7$ and $8 + 38 = 46$.

_____ 13. Water is wet or water is not wet.

_____ 14. 'George Washington was our first President' implies 'Abraham Lincoln was the 16th President.'

_____ 15. A cat is a four-legged animal or a cat is a three-legged animal.

_____ 16. 9 is a multiple of 4 or 9 is a multiple of 5.

_____ 17. If $3 + 5 = 4$, then $3 \div 9 = 8/9$.

_____ 18. $x = 3$ if and only if $4x = 12$.

_____ 19. $5 > 3$ or $5 = 2 + 3$.

_____ 20. 7 is a prime if and only if 11 is composite.

_____ 21. If this stone is released then it falls to the ground.

_____ 22. Not all birds have wings.

_____ 23. It is not the case that Washington was not the first U. S. President.

_____ 24. Shakespeare is not the author of "Waverly".

_____ 25. It is not either raining or snowing.

Notice that by removing certain connective words like 'if ... then,' 'or,' 'a,' 'if and only if,' 'it is not the case that,' and 'not' reduces these sentences into smaller sentences. These words connected to the smaller sentences are called sentential connectives.

The sentence formed by combining two sentences by the connective 'if ... then ...' is called a conditional.

1. Which of the sentences above are conditionals?

The sentence formed by combining two sentences with 'and' is called a conjunction.

2. Which of the sentences above are conjunctions?

The sentence formed by combining two sentences with 'or' is called a disjunction.

3. Which sentences are disjunctions?

The sentence formed by combining two sentences with 'if and only if' are called equivalences.

4. Which of the sentences are equivalences?

The sentence formed by combining 'not' or 'it is not the case that' with a sentence is called a negation.

5. Which of the sentences is a negation?

Introduce the symbols for the connectives:

$\rightarrow$ for if ... then

$\wedge$ for 'and'

$\vee$ for 'or'

$\leftrightarrow$ for 'if and only if'

$\sim$ for 'not'

If a sentence contains no connective it is a simple <u>atomic</u> sentence. A sentence made up of atomic sentences with connectives is a molecular sentence.

Record information from an examination of each conjunction in a table.

	p	q	p ∧ q
1.	t	t	t
2.	t	f	f
3.	f	t	f
4.	f	f	f

Tell students to summarize their findings concerning the truth values of a conjunction. (A conjunction is true only when both components are true. 'And' can be interpreted as both.)

Here is the truth table that defines the 'V' connective:

	p	q	p ∨ q
1.	t	t	t
2.	t	f	t
3.	f	t	t
4.	f	f	f

What does a 'V' statement say, i.e. when is it true? Lead the students to say 'at least one of either p or q.' This is approximated by the English: p or q.

Questions will probably be posed by students on the two meanings of 'or'. The teachers may explain that mathematicians use 'or' to mean at least one; hence a disjunction is true when at least one of the components is true. Several English examples of the exclusive 'or' should be found by students. Students will probably be interested in exclusive 'or' and should be encouraged to give examples in which the intended meaning can only be the exclusive 'or' (denote by "$\underline{\vee}$"), for example, Jane is a freshman or a sophomore.

A truth table for an exclusive 'or' :

	p	q	p $\underline{\vee}$ q
1.	t	t	f
2.	t	f	f
3.	f	t	t
4.	f	f	f

The exclusive disjunction is true when exactly one component is true.

Here is the truth table that defines the '→' connective:

	p	q	p → q
1.	t	t	t
2.	t	f	f
3.	f	t	t
4.	f	f	t

The conditional, $p \rightarrow q$, is true on lines 1, 3 and 4. How can this be said in English? On lines 3 and 4 where p is false we see q can be either true (line 3) or false (line 4) without affecting $p \rightarrow q$. So let us see on lines 1 and 2 what happens if p is true, and of course $p \rightarrow q$ is true. That's only on line 1. So, if p is true then q is true for true conditionals. We can thus read it, 'if p then q'.

Does it seem strange to call a conditional like: "If I release this stone then it falls to the ground" true when I do not release this stone? When I do not release the stone we still believe it is true that <u>if</u> I release this stone then it falls to the ground.

The last column of this truth table defines the connective '$\leftrightarrow$'.

	p	q	$p \rightarrow q$	$q \rightarrow p$	$p \rightarrow q$ and $q \rightarrow p$	$p \leftrightarrow q$
1.	t	t	t	t	t	t
2.	t	f	f	t	f	f
3.	f	t	t	f	f	f
4.	f	f	t	t	t	t

Notice that '$p \rightarrow q$ and $q \rightarrow p$' has identically the same truth table as '$p \leftrightarrow q$'. This shows the equivalence is itself equivalent to the conjunction of the conditional and its converse, two conditionals. For this reason it is often called a bi-conditional.

You can translate $p \leftrightarrow q$ as:

p is true if and only if q is true,

p is necessary and sufficient for q, or

q is true if and only if p is true.

Any two sentences with always the same truth values are called equivalent. Finally the students should be able to readily fill out the truth table for the negation.

p	~p
t	f
f	t

Have the students construct a truth table for these more complex formulas.

	p	q	~p	~q	pVq	~pA~q	~(pVq)
1.	t	t	f	f	t	f	f
2.	t	f	f	t	t	f	f
3.	f	t	t	f	t	f	f
4.	f	f	t	t	f	t	t

How do the truth values of the last two columns compare? They are equivalent sentences. Here is an English example:

~ p A q: It is not raining and it is not snowing

~ (pVq): It is not either raining or snowing.

The equivalence of these two sentences is an example of DeMorgan's Law.

p	q	~p	~q	p∨q	~(p∨q)	~p∨q	p→q	~p→q	p↔q	~(p↔q)
t	t									
t	f									
f	t									
f	f									

p	q	~p	~q	p∨q	~(p∨q)	~p∨q	p→q	~p→q	p↔q	~(p↔q)
t	t									
t	f									
f	t									
f	f									

p	q	~p	~q	p∨q	~(p∨q)	~p∨q	p→q	~p→q	p↔q	~(p↔q)
t	t									
t	f									
f	t									
f	f									

To students:

(1) Find all equivalent sentences.

(2) Find all forms of negations of statements.

The last two columns of the above table are quite different from the preceding ones. They require information not found in the table. The final column had all T's except for one f. The other column had all t's. These columns are forms of arguments in logic. Arguments in logic are valid or invalid according to their form (irrespective of what may be true or false in reality).

The last three formulas on the bottom now have a striking property; they are true regardless of the truth value of the atomic sentences. Such a sentence is called a tautology. Ask students to find other tautologies in this table.

A set of premises with a proposed conclusion is called an argument. If the conclusion follows logically from the premises, the argument is said to be valid; otherwise it is invalid. Determine for each of the following arguments whether it is valid or invalid.

1. If prices are going still higher, inflation is surely coming.
   Prices are going still higher.
   ∴ Inflation is surely coming.
2. If it is raining, I will wear my raincoat.
   It is raining.
   ∴ I will wear my raincoat.

3. If he is tall, he is thin.

   He is thin.

   ∴ He is tall.

4. If he is a sailor then he goes to sea.

   He does not go to sea.

   ∴ He is not a sailor.

5. If today is Thursday, then yesterday was not Tuesday.

   Yesterday was Wednesday.

   ∴ Today is Thursday.

6. If it is green, then it is grass.

   It is green.

   ∴ It is grass.

7. If Los Angeles is not in California, then golf balls are not sold in Chicago.

   Golf balls are sold in Chicago.

   ∴ Los Angeles is in California.

8. If the week has seven days, then fish have wings.

   If fish have wings then there are fifty-two weeks in a year.

   ∴ If the week has seven days, then there are fifty-two weeks in a year.

9. If he gambles, then he loses money.

   He does not gamble.

   ∴ He does not lose money.

10. The given triangles are similar.

    ∴ The triangles are mutually equiangular.

    If the given triangles are mutually equiangular, then they are similar.

11. If angle A = Angle B, then AC = BC.

    AC = BC

    ∴ Angles A = Angle B

12. If it is a wab, then it is a yud.

    It is a wab.

    ∴ It is a yud.

13. It is round, if it is blue.

    If it is blue, then it holds water.

    ∴ It it is round, then it holds water.

14. If Delaware is larger in area than Texas, then Australia is larger than the U. S.

    Delaware is larger in area than Texas.

    ∴ Australia is larger than the U. S.

15. The days are becoming shorter.

    "The nights are becoming longer" implies that "the days are becoming shorter".

    ∴ The nights are becoming longer.

16. If X passes the final examination, then he will pass the course.

    X will pass the course.

    ∴ X will pass the final examination.

17. If wages are not raised, then buying decreases.

    If there is a depression wages cannot be raised.

    ∴ If there is a depression then buying cannot increase.

18. If my father gives me a dollar, then I'll get a milkshake.

    ∴ My father gave me a dollar.

    I get a milkshake.

19. If it is a Ford, it burns oil.

    It is not a Ford.

    ∴ It does not burn oil.

After these arguments have been studied intuitively, the students will be ready for a formal approach to validity and invalidity.

First the argument should be translated into symbolic notation. This in effect analyzes the argument into its logical structure and suppresses all referential content.

Second, a truth table is prepared for the premises and the conclusion, and it is examined for proof of validity or invalidity applying the following definition:

> An argument is valid if and only if, on any row of the truth table where the premises are all true, the conclusion is also true, i.e., there is no row on which the premises are all true and the conclusion false.

The students should be able to suggest, therefore, an argument that is invalid if there is any one line with true premises and a false conclusion.

Let us examine the exercise argument 2.

Let R represent "It is raining"

W represent "I will wear my raincoat"

Our argument is now:

1. $R \rightarrow W$

2. R

Conclusion: W

	R	W	$R \rightarrow W$
	2.	Conclusion	1.
1.	t	t	t
2.	t	f	f
3.	f	t	t
4.	f	f	t

Only on row 1 are both premises true, and there the conclusion is also true.

also true.

Argument 10 can be symbolized:

1. $G \rightarrow L$

2. $\sim G$

Conclusion: $\sim L$

	G	L	$\sim G$	$G \rightarrow L$	$\sim L$
			2.	1.	Conclusion
1.	t	t	f	t	f
2.	t	f	f	f	t
3.	f	t	t	t	f
4.	f	f	t	t	t

On rows 3 and 4 both premises are true. But on row 3, the conclusion is false. Therefore, the argument is invalid. Row 4 can not save it; a single counter example proves invalidity.

The students can now give a formal analysis of several of the arguments. It will be particularly interesting for any arguments for which there was originally disagreement among class members as to whether it is valid or invalid.

Several of the students may soon realize that the whole truth table is unnecessary if they are clever enough to find a single line proving invalidity. Of course for validity the entire table is necessary; any omitted line might demonstrate invalidity.

An alternate way of using a truth table to analyze validity is to construct the conditional corresponding to the argument, as follows:

(conjunction of the premises) → (conclusion)

Now the argument is valid if and only if the corresponding conditional is a tautology.

If the students are lead to see the exact parallel between "a conditional is false only when antecedent is true and consequent false" and "an argument is invalid only if there is a line in its truth table where premises are true and conclusion false" they may themselves come up with the tautologicality of the conditional corresponding to a valid argument.

Now the class can return to the problems pages 10 - 12. Construct the corresponding conditionals and test the validity by determining whether or not they are tautologies.

Then study the standard rules of derivation (affirm the antecedent, deny the consequent, and deny a disjunct) and the standard fallacies (deny the antecedent, affirm the consequent, affirm a disjunct) pages 13 - 16, starting "The students should discover the various standard patterns of valid and invalid arguments," verifying them with truth tables or counter examples. (Modus Ponens)

1. Affirming the antecedent

(1) $p \rightarrow q$

(2) p

Conclusion: q

Let us consider some examples of the use of this rule in deriving conclusions from premises.

Premise 1. If he is on the football team, then he is at the practice field now.

Premise 2. He is on the football team.

Conclusion: He is at the practice field now.

Another example of the use of "Modus Ponens".

Premise 1. If it is not cold, then the lake will not be frozen.

Premise 2. It is not cold.

Conclusion: The lake will not be frozen.

This example can be symbolized in the following way, where p is the sentence "It is cold" and q is the sentence "the lake will be frozen".

$$\begin{array}{l} \sim p \rightarrow \sim q \\ \sim p \\ \therefore \sim q \end{array}$$

2. Premise 1. If it shines by its own light, then the heavenly object is a star.

   Premise 2. The heavenly object is not a star.

   Conclusion: Therefore, it does not shine by its own light.

The above example may be symbolized in this way.

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \therefore \sim p \end{array}$$

This argument is called "deny the consequent" (Modus Tollens). It is a valid argument.

Let us look at another example. The conditional sentence is: If it is morning, then the sun will be in the east. (Note that the second premise denies the consequent of the conditional and the conclusion denies the antecedent)

We deny its consequent:

The sun is not in the east.

Thus we can deny its antecedent:

Therefore, it is not morning.

3. $p \vee q$ the name of this argument is Deny the Disjunct (Modus Tollendo Ponens).

   $\sim p$

   $\therefore q$

4. $p \rightarrow q$ This valid argument is called "hypothetical syllogism"

$q \rightarrow r$

$\therefore p \rightarrow r$

5. $p \rightarrow q$ This is another valid argument. It requires the new rule, Simplify the Conjunction.

$q \rightarrow (r \vee s)$

$p \wedge \sim r$

$\therefore s$

(1)	$p \rightarrow q$	p
(2)	$q \rightarrow (r \vee s)$	p
(3)	$p \wedge \sim r$	p
(4)	p	Simplify conjunction 3
(5)	q	Affirm antecedent 1, 4
(6)	$r \vee s$	Affirm antecedent 2, 5
(7)	$\sim r$	Simplify conjunction 3
(8)	s	Deny a disjunct 6, 7

This proof required several steps. The result of one is used in others, so a full formal proof is needed to keep track.

The following three arguments are not valid.

6. $p \rightarrow q$ This argument is invalid due to the affirming of the consequent.

q

$\therefore p$

7. $p \rightarrow q$ — The fallacy of denying the antecedent causes this argument to be invalid.

   $\sim p$

   $\sim q$

8. $(p \vee q)$ — This is another invalid argument by affirming a disjunct.

   p

   $\therefore q$

Students should be led to realize that validity and invalidity are matters of form. By constructing truth tables they will always be able to establish whether or not an argument is valid.

Example 1

If the clock has stopped, then the class has started.

The clock has stopped.

$\therefore$ The class has started.

p -- the clock has stopped.

q -- the class has started.

$p \rightarrow q$

p

$\therefore q$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

The given argument is valid because it has no counter examples of the inference pattern, i.e., the corresponding conditional is a tautology.

If the sun rises in the east, then early morning shadows cannot point eastward.

Early morning shadows cannot point eastward.

The sun rises in the east.

p -- If the sun rises in the east.

q -- Early morning shadows cannot point eastward.

$p \rightarrow q$

q

$\therefore p$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$[(p \rightarrow q) \wedge q] \rightarrow p$
t	t	t	t	t
t	f	f	f	t
f	t	t	t	f
f	f	t	f	t

This argument is invalid. The students should discover the counter example to the argument on the third line. If the student is clever enough, he will realize that he need not consturct the rest of the table; the third line is enough.

An Alternative to the t and f Notation

We have used the set of symbols {t, f, ∨, ∧, ~} to construct the truth tables. It may be worthwhile to point out that other symbols may also be used to achieve the same purpose, particularly the binary operations of logical product, "AND", and logical sum, "OR", and the unary operation of negation represented by ~ . The ~ may be used with the first two operations to form "NOT AND" (nan) and "NOT OR" (nor). Usually the symbol "+" (plus) is used for "OR" and "." (dot) is used for "AND". The two binary numbers 0 and 1 are used for "False" and "Truth" respectively. This representation is easy for students to understand because the "OR" operation works similarly to the addition and the "AND" operation similarly to the multiplication on the binary numbers 0 and 1. Furthermore, the latest trend in the study of mathematics is to look for common structures and differences. Axioms and properties of the decimal system with which the students are already familiar are used as a basis for studying these structures. These structures in turn may be used for the study of Logic.

Familiarity with basic truth tables using the set of symbols, {f, t, ∨, ∧, ~} will make it easy to construct tables using {0, 1, +, ., ~}, Consider the following tables:

p	q	p + q	q + p
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

"OR" Table

p	q	p . q	q . p
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

"AND" Table

Hence, the operations "OR" and "AND" are commutative. Note that $1 + 1 = 1$. In the decimal system of real numbers $1 + 1 \neq 1$. Thus the students need to realize that they are working in a different system.

Also, consider the following tables:

p	q	r	(p+q)	(p+q)+r	(q+r)	p+(q+r)	(p.q)	(p.q).r	(q.r)	p.(q.r)
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	0	0	0	0
0	1	0	1	1	1	1	0	0	0	0
0	1	1	1	1	1	1	0	0	1	0
1	0	0	1	1	0	1	0	0	0	0
1	0	1	1	1	1	1	0	0	0	0
1	1	0	1	1	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1	1

From this table, we see that the associative laws hold for the operations "OR" and "AND".

Now, consider the following tables:

p	q	r	q+r	p.(q+r)	p.q	p.r	(p.q + p.r)
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

From this table, we see that

$$p.(q+r) = p.q + p.r.$$

p	q	r	(q.r)	p+(q.r)	(p+q)	(p+r)	(p+q) . (p+r)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Notice that $p+(q.r) = (p+q) . (p+r)$. However, in the decimal (real number) system $p+(q.r) \neq (p+q) . (p+r)$. This is another point of difference.

Again, consider:

p	q	(p+q)	~(p+q)	~p	~q	(~p) . (~q)
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

This table shows that ~(p+q) = (~p) . (~q) which is not true in the decimal system.

Consider the following table:

p	q	p.q	~(p.q)	~p	~q	(~p) + (~q)
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

This table shows that ~(p.q) = (~p) + (~q), which is not true in the decimal system.

These two results:

$$\sim(p+q) = (\sim p) \cdot (\sim q) \quad \text{and}$$

$$\sim(p.q) = (\sim p) + (\sim q)$$

are known as De Morgan's Laws. These two relationships are referred to as "The Duality Principle".

Quantifiers

Certain statements involve words that indicate quantity such as "all","some," "none," "one," and "every". Since such words indicate quantity they are called quantifiers.

Examples:

1. All birds are small.
2. Some men are tall.
3. No balloon is perfectly round.

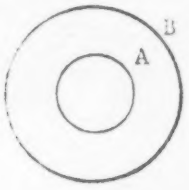
Arguments involving these words may be analyzed in terms of sets and Venn diagrams. Illustrate the following statements to the class:

1. All A is B.

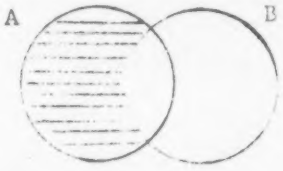
   A is a subset of B.

   $A - B = \phi$.

   These are equivalent statements. They can be represented by:



or by



In the second diagram the horizontal shading indicates that any part of A you might think is outside, is empty. So all of A is in B.

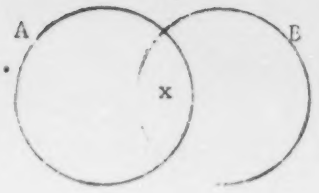
2. No A is B,

   $A \cap B = \phi$.

or

3. Some A's are B's.

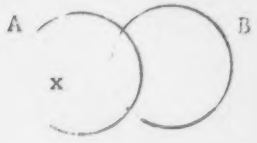
$A \cup B \neq \phi$.



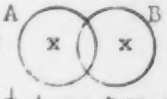
(The cross indicates there is something in $A \cap B$; it is not empty.)

4. Some A's are not B's .

$A - B \neq \phi$.

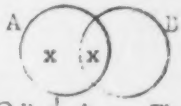


5. How would you read the following?

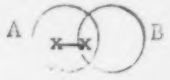


(Answer: $A - B \neq \phi$, and $B - A \neq \phi$ -- two statements)

6. How would you read this?



(Answer: $A - B \neq \phi$ and $A \cap B \neq \phi$ -- The crosses are separate and therefore they make separate statements. To diagram the statement $A \neq \phi$ linked crosses are used*-- link crosses make a single statement)

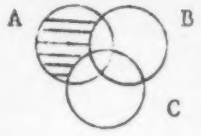


7. How do you read this?

(Answer: $(A-B) \cup (B-A) \neq \phi$)

* It says the union of the two parts of A is non-empty, but either part could be empty as long as the other is not empty.

8. For 3 sets or 3 properties we use a Venn Diagram of 3 linked circles:



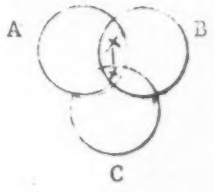
This is read: All A's are either B's or C's

or: $(A - B) - C = \Phi$

or: No A is non B and non C

Note: No claim is made for the regions containing neither shading nor crosses. The diagram is always read by reading the shaded or crossed regions, not the blank ones. So, 'all A's are either B or C' does not claim there are any A's -- there are no crosses.

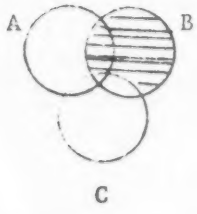
9. 'Some A's are B's'



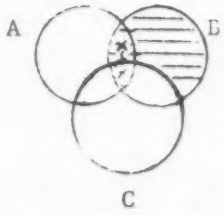
and

'all B's are C's'

(i.e. no B's outside of C.)



combine to imply



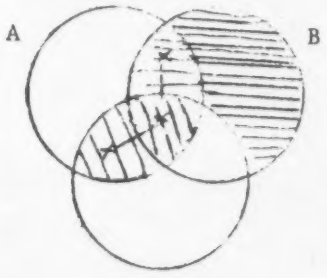
The non-C part of $A \cap B$ contains both a cross and shading. Does this involve contradictory claims that it is both empty and not empty? No, the cross is linked to another. The chain says the whole region $A \cap B$ is non-empty. The shading shows it is not in the upper part of the region; therefore we know the middle of the diagram is non-empty. It can be read, "Something is simultaneously A, B and C." Or, since we know by the first premise that some A's are B's, the new information learned from the combined diagram is simply, some A's are C's.

8. Make a single diagram for the three sentences:

   (a) Some A is either B or C. (by the linked crosses)

   (b) All B's are C's. (by horizontal shading)

   (c) No A are C. (by slanting shading)



What strange condition occurs in this diagram?

(Answer: $(B \cup C) \cap A$ is both empty and non-empty. This is logically impossible. We therefore say that sentences (a), (b) and (c) are inconsistent.

The instructor should explain to the students that an analysis of mathematical sentences involving quantifiers indicates that the two standard symbols of quantification are $\forall$ and $\exists$. $\exists$ stands for "for all", "for every", "for each" and stands for "some", or "these exists".

Arguments involving these quantifiers may be valid or invalid but these aspects cannot be checked by the use of a truth table. Derivation by valid rules out the study of models are the appropriate methods.

Exercises

Find out whether the following arguments are valid and invalid by using Venn Diagrams.

1. All A's are B's.

   All B's are C's.

   $\therefore$ All A's are C's.

2. All roses are flowers.

   This is a flower.

   $\therefore$ This is a rose.

3. No undergraduates have B.A. Degrees.

   No Freshmen have B.A. Degrees.

   $\therefore$ Freshmen are undergraduates.

4. Some isosceles triangles are equilateral.

   All equilateral triangles are equiangular.

   $\therefore$ Some isosceles triangles are equiangular.

5. Some A's are B's.

   All B's are C's.

   ∴ Some A's are C's.

6. All A's are B's.

   No B's are C's.

   ∴ No A is C.

Ask the students to complete each of the following syllogisms. Use a Venn diagram and read off the new information for the conclusion.

1. If this is a soli, then this a proli.

   If this is a proli, then this is an umpi.

   ∴ _____

2. All Virginians are residents of the United States.

   All Norfolkians are residents of Virginia.

   ∴ _____

3. All equilateral triangles are equiangular.

   No equiangular triangle is a right triangle.

   ∴ _____

4. All squares are rectangles.

   All rectangles are parallelograms.

   ∴ _____

5. All A's are B's.

   This is an A.

   ∴ _____

6. All cats have whiskers.

   Tobby is a cat.

   _____

Indirect Proof

A proof is a demonstration that the conclusion of an argument follows logically from the premises. It may be a truth table showing that if the premises are true, the conclusion is also true; it may be a derivation of the conclusion from the premises using valid rules of derivation. For quantifier logic, the unit does not go into the rules for derivation; we use Venn Diagrams carefully analyzed.

A conditional proof derives a conditional of the form $p \rightarrow q$ (if p then q). Assuming p and the premises derive q, this shows that if we had p then we could derive q. That is just what the conditional says: if p then q.

Our final method of proof is called indirect proof. The method is as follows: Assume the denial of the conclusion as an added premise. Derive a contradiction. Assert the conclusion (denial of added premise) by indirect proof.

To prepare for indirect proof be sure that the students are aware that any sentence and its negation have opposite truth values -- one true and the other false. Thus their conjuction, in the form $p \wedge \sim p$ is always false. Any conjunction of this form is called a contradiction.

Here is an example of an informal indirect proof:

With the premises that (1) Every integer is either odd or even, (2) For any even integer x it is possible to find an integer z such that $x = 2z$, and any number in the form 2z is even, (3) Product of two integers is an integer, and (4) xy is odd, give an indirect proof that both x and y are odd.

Proof: Suppose that not both x and y are odd. Then by (1) at least one of them is even, say x. Then for some integer z, $x = 2z$. So, $xy = 2zy$. This shows that xy is two times some number and therefore is even. So it is not odd. Thus we get the contradiction: xy is odd and xy is not odd. Similarly the contradiction could be derived if we had taken y as even. By indirect proof we can thus deny our supposition and conclude: Both x and y are odd.

The student should clearly see that the denial of the conclusion to be proved was added as an extra premise and a contradiction (any contradiction will do) was derived. The conclusion (the denial of the added premise) was then immediately asserted as proved by indirect proof.

Here is another example:

(1) Either Mary is screaming or John is quarreling, (2) If John is quarreling then there is no peace. (3) If there is peach then Mary is not screaming. Prove: There is no peace.

To concentrate on logical form it is best to symbolize:

Prove: ~p

(1)	s ∨ q	premise
(2)	q → ~ p	premise
(3)	p → ~ s	premise
(4)	p	assumption for indirect proof
(5)	~q	deny consequent 2, 4
(6)	s	deny a disjunct 1, 5
(7)	~p	deny consequent 3, 6
(8)	p ∧ ~ p	conjoin 4, 8
(9)	~p	indirect proof 4, 8

Note that the proof was not complete on line 7 because that was derived assuming p. To be free of the assumption the indirect proof step is necessary. By a different choice of steps 5, 6 and 7, line (8) might have been q ∧ ~ q or s ∧ ~ s. It does not matter which as long as a contradiction is obtained.

The steps of an indirect proof are:

(1) Assume as added premise the denial of what is to be proved.

(2) Derive a contradiction.

(3) Assert the denial of the added premise (the assumption) by indirect proof.

Sometimes a conditional like p → q is derived by deriving its contrapositive, ~ q → ~ p, to which it is equivalent. This is sometimes referred to as a kind of indirect proof but it is probably better to recognize it as simply a conditional proof with a twist.

Note also that when an assumption is addes as a working premise the formulas in the formal proof were indented a little there and on the lines following. This keeps track of the fact that the added premise is being depended on. Only by indirect proof or conditional proof can one move back to the main proof.

At any point in a proof another assumption can be added. Here is an indirect proof within a conditional proof:

Prove: $r \rightarrow \sim s$

(1)	$\sim t$	premise
(2)	$(r \wedge s) \rightarrow t$	premise
(3)	$\sim q \vee t$	premise
(4)	r	assumption
(5)	s	assumption
(6)	$r \wedge s$	conjoin 4, 5
(7)	t	affirm antecedent 2, 6
(8)	$t \wedge \sim t$	conjoin 7, 1
(9)	$\sim s$	indirect proof 5, 8
(10)	$r \rightarrow \sim s$	conditional proof 4, 9

Line (4) is an automatic step -- antecedent of the conditional $r \rightarrow \sim s$ that we want to prove by conditional proof. With r, we now need to derive $\sim s$, the consequent. With no obvious way to get at it, we apply the maxim, "When in doubt try an indirect proog". So the denial of $\sim s$, i.e., s, is the next assumption added on line (5), indented a second time.

Here is another one the better students should be able to do.

Prove: $t \rightarrow m$

(1) p

(2) $(t \rightarrow s) \vee \sim p$

(3) $\sim s$

(Note: m does not appear in the premises so it could be any formula)

There is no automatic technique for deriving a conclusion. The rules do not tell which way to proceed. That is a matter of strategy obtained by experience and ingenuity. A great deal is to be learned by studying proofs others have made. Examine the different proofs constructed by members of the class.

It is poor strategy to work a long time unsuccessfully on a proof without looking for a counter example to prove it invalid.

Exercises:

Derive, or prove invalid.

1. Prove: r

   (1) p

   (2) $q \rightarrow \sim p$

   (3) $\sim q \rightarrow r$

2. Prove: r

   (1) $r \vee \sim s$

   (2) s

3. Prove: ~ s

   (1) p

   (2) q → ~ p

   (3) ~ q → s

4. Prove: r

   (1) ~ p

   (2) (~ p ∨ q) → r

5. Prove: ~ r → t

   (1) ~ t → ~ s

   (2) s ∨ r

6. Prove: s → r ∧ s

   (1) r

7. Prove: ~ q

   (1) t → q

   (2) ~ t ∨ r

   (3) ~ r

8. Prove: s → s

   with no given premises.

9. Prove: r → ~ q

   (1) ~ r ∨ ~ s

   (2) q → s

10. Prove: q → (r → s)

    (1) (q ∧ r) → s

    this is a conditional proof

    within a conditional proof.

11. Prove: m

    (1) ~ m → ~ r

    (2) q → m

    (3) r ∨ q

12. Prove: ~ (r ∧ ~ r)

    no premise given.

13. Prove: t

    (1) ~ (p ∨ q)

    (2) p ∨ r

    (3) t → r

Proofs

Discuss the following proofs and decide which are valid and which are invalid.

a) Proof that two lines are parallel.

Two lines, which we may call L_1 and L_2, are parallel, because if a line is drawn cutting both of them, the alternate interior angles formed will be equal and equal angles determine parallel lines. We know that the angles are equal because the angles formed by parallel lines are equal.

b) Proof that a line segment is the shortest distance between two points.

The line AB is shorter than any two different paths, say AC + CB through a point C not on AB, because two sides of a triangle are greater than the third. The distance AC + CB is greater because any broken line between two points is longer than the straight line joining them.

c) Proof that a rhombus is a plane figure.

A rhombus is a parallelogram.

A parallelogram is a quadrilateral.

Any quadrilateral is a plane figure.

Therefore, a rhombus is a plane figure.

d) Proof that a square is a parallelogram.

A square is a rectangle. All rectangles are parallelograms, and so a square is a parallelogram.

e) Proof that a traffic light is a wast of money.

The traffic light costs money for operation and repair. There have been no accidents at the intersection since the light was installed. Therefore, the traffic light is an unnecessary wast of money.

f) Proof that 8 > 2.

1. 8 > 7
2. 7 > 6
3. 6 > 5
4. 5 > 4
5. 4 > 3
6. 3 > 2
7. Therefore, 8 > 2.

g) Proof that Mark is not in class.

Proof (indirect) Either Mark is in class or Mark is not in class.

Assume Mark is in class.

Then, if I look at everyone in class, I will see him.

I look around.

I see him.

But I do not see him. Contradiction.

The assumption which led to this contradiction is false.

Therefore the negation of the assumption is true which is to say that Mark is not in class.

h) Proof that 15 is not an even number.

Proof (indirect) Either 15 is even or 15 is not even.

Assume 15 is an even number.

Then 15 = 2 ☐ where ☐ represents a natural number.

But if 15 = 2 ☐ then ☐ represents 7 1/2 which is not a natural number. Thus ☐ represents a natural number and ☐ represents a number which is not natural.

Contradiction.

The assumption which led to the contradiction is false. Thus, the negation of the assumption is true or the statement 15 is not even, is true, as required.

i) Proof that we are not traveling north.

Proof (indirect) Either we are going north or we are not going north.

Assume we are traveling north.

Then we deduce that the morning sun would be on my right arm.

If the sun is on my right arm, then it fill feel warm.

My right arm feels warm.

But my right arm does not feel warm. (My left one does.)

Contradiction

Hence the assumption which led to this contradiction is false, which means that the negation of the assumption is true.

Which means, "We are not going north."

j) Proof that 2 = 1.

Let $a = b$.

then $a^2 = a.a = a.b$

and $a^2 - b^2 = a.b - b^2$

so $(a - b)(a + b) = (a - b)\,b$

and dividing by $a - b$ we have $a + b = b$

so $b + b = b$

or $2b = 1b$

and dividing by b we have $2 = 1$.

EXERCISES

Part I

1. Let m and n be positive integers. Prove that if nm is odd, then m and n are both odd.
2. The board of govenors of the country club removed from club committees all members whose dues were in arrears. Which of the following statements about club member Jones reflect this action by the board?
   a) Jones is on a committee if his dues are paid up.
   b) In order for Jones to be on a committee it is necessary that his dues be paid up.
   c) For Jones' dues to be in arrears, it is sufficient that he not be on a committee.
   d) If Jones is not on a committee, then his dues are in arrears.
   e) Jones is on a committee only if his dues are paid up.
   f) For Jones' membership on a committee it is a necessary condition that his dues be paid up.
   g) Jones is not on a committee only if his dues are in arrears.
   h) If Jones is on a committee, then his dues are paid up.
3. To win the election the Republican must carry New York.
   a) Write this in the form "If _____ then _____.
   b) Give the contrapositive of the statement in part a).
   c) Give the converse of the statement in part a).
   d) Give the contrapositive of this converse statement.

4. Show that, for a given conditional statement, the converse of its contrapositive is the contrapositive of its converse.

5. We have seen that $(p \to q) \leftrightarrow (\sim p \vee q)$.

   a) Write the contrapositive of $p \to q$ in terms of $\sim$ and $\vee$.

   (b) Write the converse of $p \to q$ in terms of $\sim$ and $\vee$.

6. Show that $\sim (p \to q) \leftrightarrow (\sim q \wedge p)$.

7. Using Exercise 6, write the contrapositive of $p \to (q \to r)$.

8. Use Exercise 7 to prove the following:

   If $x > 0$, then $x^2 = 4$ implies $x = 2$.

9. Show that $[(p \to q) \to (r \to s)] \leftrightarrow [(\sim r \vee s) \vee (p \wedge \sim q)]$.

10. A man about to be electrocuted is given a chance for his life. In the execution chamber are two chairs labeled 1 and 2, and a jailer. One chair is electrified, the other is not. The prisoner must sit in one of the chairs but before doing so he may ask the jailer one question which the jailer must answer yes or no. The jailer is a consistent liar or else a consistent truth teller, but the prisoner does not know which. Knowing that the jailer deliberately lies or faithfully tells the truth, what question should the prisoner ask?

11. Do you know that Sam, the campus barber, regularly shaves all (and only those) men of the campus who do not shave themselves? Does Sam shave himself?

12. Five suspects were quizzed in connection with the "Slaten Pizza Case". Their statements were as follows:

    A: "C and D are lying".

    B. "A and E are lying".

    C: "B and D are lying".

    D. "C and E are lying".

    E. "A and B are lying".

    Who is lying?

13. Four freshmen boys, Albert, Bob, Carl, and Dan and four freshmen girls, Edna, Fay, Gail and Hilda, are each in love with one of of the other and sad to say, in no case is their love requitted. Albert loves the girl who loves the boy who loves Edna. Fay is loved by the boy who is loved by the girl loved by Bob. Carl loves the girl who loves Dan. If Bob is not loved by Gail, and the boy who is loved by Hilda does not love Gail, who loves Albert?

14. The engineer, brakeman, and fireman on a train are named Smith, Jones and Robinson, but not necessarily in that order. On the train are three passengers by the names of Mr. Smith, Mr. Jones, and Mr. Robinson.

    a) Mr. Robinson lives in Washington.

    b) The brakeman lives in Arlington.

    c) Mr. Jones long ago forgot all the Algebra he learned in high school.

d) The passenger whose name is the same as the brakeman's lives in Maryland.

e) The brakeman and one of the passengers, a distinguished mathematician, attend the same church.

f) Smith beat the fireman at poker.

Who is the engineer?

Part II

Use Venn Diagrams to check the validity of the following:

1. No equilateral triangle is a right triangle.

   All equiangular triangles are equilateral.

   Therefore, no equiangular triangle is a right triangle.

2. Some parallelograms are squares.

   All squares are rectangles.

   Therefore, some parallelograms are rectangles.

3. No prime integers greater than 2 are even.

   x is a prime integer greater than 2.

   Therefore, x is not an even integer.

4. Some primes are integers divisible by 2.

   All integers divisible by 2 are even integers.

   Therefore some primes are even integers.

5. All rectangles are parallelograms.

   Not all quadrilaterals are rectangles.

   Therefore, not all quadrilaterals are parallelograms.

Part III

Determine whether each of the following arguments is valid or invalid. If the argument is invalid, indicate a combination of truth-values which will produce a counterexample.

1. If today is Thursday, then yesterday was Wednesday.

   Yesterday was Wednesday.

   Therefore, today is Thursday.

2. If Frank or Marge needs a vacation, then Kate deserves an assistant.

   Therefore, if Frank needs a vacation, then Kate deserves an assistant.

3. If the parakeet likes celery, then Jim will buy celery.

   Therefore, the parakeet likes celery.

4. If the innkeeper explained, there is no reason for argument.

   The innkeeper explained.

   Therefore, there is no reason for argument.

5. If Los Angeles is not in California, then golf balls are sold in Chicago. Therefore, Los Angeles is in California.

6. If the cup is gold, then it is lighter than water.

   If the cup is lighter than water, then Janie can carry it.

   Therefore, if the cup is gold, then Janie can carry it.

7. If wishes are horses, beggars can ride.

   Beggars can ride.

   Therefore, wishes are horses.

8. If the week has seven days, then fish have wings.

   If there are not fifty-two weeks in a year, then fish do not have wings.

   Therefore, if the week has seven days, then there are fifty-two weeks in the year.

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